Sphericity and multiplication of double cosets for infinite-dimensional classical groups

Yury A. Neretin[1]

We construct spherical subgroups in infinite-dimensional classical groups G (usually they are not symmetric and their finite-dimensional analogs are not spherical). We present a structure of a semigroup on double cosets $L \setminus G/L$ for various subgroups L in G, moreover these semigroups act in spaces of L-fixed vectors in unitary representations of G. We also obtain semigroup envelops of groups G generalizing constructions of operator colligations.

1 Introduction

1.1. Definition of spherical subgroups. Let G be a group, $K \subset G$ a subgroup. An irreducible unitary representation ρ of G in Hilbert space H is called K-*spherical*, if there exists a unique (up to a proportionality) K-fixed vector $v \in H$. The *spherical function* of a spherical representation is given by the formula

$$\Phi(g) = \langle \rho(g)v, v\rangle.$$

Let G be a semisimple Lie group, K a compact subgroup. The pair (G, K) is called *spherical*, if any irreducible unitary representation of G contains at most one non-zero K-fixed vector. The subgroup K is said to be a *spherical* subgroup of G.

According to the Gelfand theorem [4], symmetric subgroups are spherical. In particular, maximal compact subgroups of semisimple groups are spherical.

Krämer classified [10] all spherical pairs (G, K) with simple group G, the classification was extended to semisimple groups in [13] and [2]. All such pairs can be produced from symmetric pairs (G, K) by a minor diminishing of the subgroup K or small enlargement of G. On spherical functions for non-symmetric pairs, see the work of Knop [8].

1.2. Infinite-dimensional classical groups. We consider Hilbert spaces l_2 over $\mathbb{R}$, $\mathbb{C}$, or quaternions $\mathbb{H}$. We call a matrix g, whose elements are contained in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *finite* if the matrix $g - 1$ has only finite number of non-zero elements. We define *infinite-dimensional classical groups* $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$, $\mathrm{GL}(\infty, \mathbb{R})$, $\mathrm{Sp}(2\infty, \mathbb{R})$, $\mathrm{U}(\infty, \infty)$, ... as groups of finite matrices satisfying the usual identities. All such groups $G(\infty)$ are inductive limits of corresponding finite-dimensional groups $G(n)$,

$$\mathrm{Sp}(2\infty, \mathbb{R}) = \varinjlim \mathrm{Sp}(2n, \mathbb{R}), \qquad \mathrm{U}(\infty, \infty) = \varinjlim \mathrm{U}(n_1, n_2), \qquad \dots$$

[1] Supported by the grant FWF, P22122.

We suppose that $G(n)$ are canonically embedded to $G(\infty)$ and act to initial basis vectors. If we consider groups of type $\mathrm{O}(\infty,\infty)$ etc. or products of groups, then we regard n in $G(n)$ as an integer vector $n=(n_1,\dots)$.

1.3. Heavy groups. The orthogonal group $\mathrm{O}(\infty)$, the unitary group $\mathrm{U}(\infty)$, and the quaternionic unitary group $\mathrm{Sp}(\infty)$ play the same role as compact groups in finite-dimensional theory. We call them *heavy.* For a heavy group K, we denote by K^α the stabilizer of initial α basis elements in the corresponding ℓ_2. We write elements of K^α as block $(\alpha+\infty)\times(\alpha+\infty)$ matrices of the form $\begin{pmatrix}1&0\\0&h\end{pmatrix}$, where $h\in K$.

Let ρ be a unitary representation of a heavy group in a Hilbert space H. Denote by $H^{[\alpha]}$ the space of K^α-fixed vectors.

Theorem 1.1 [23] *The following conditions are equivalent:*

— ρ *is continuous with respect to the weak topology on* K*;*

— ρ *is continuous with respect to the uniform topology on* K*;*

— $\cup H^{[\alpha]}$ *is dense in* H*;*

— ρ *admits an extension to a weakly continuous representation of the semigroup of operators with norm* $\leqslant 1$ *in* ℓ_2[2].

If ρ *is irreducible, this is equivalent to*

— $H^{[\alpha]}\ne 0$ *for some* α.

We call such representations *continuous.* Description of continuous representations is contained in [23], all such representations can be obtained by tensor operations from the simplest representation of K in ℓ_2.

Finite products of heavy groups also will be called *heavy.* If $K=K_1\times\dots\times K_p$, then we assume $K^\alpha:=K_1^{\alpha_1}\times\dots\times K_p^{\alpha_p}$, now $\alpha=(\alpha_1,\dots,\alpha_p)$ is a multi-index. *Irreducible representations of a heavy group* $K=K_1\times\dots\times K_p$ *are tensor products of representations of* K_j.

1.4. Olshanski's list. The following list consists of pairs (G,K)= (classical group, symmetric heavy subgroup). Explicits form of embeddings $K(\infty)\to$

[2]We call such operators *contractions.*

$G(\infty)$ are sufficiently obvious.

Pairs of noncompact type	Pairs of compact type
$\bigl(\mathrm{GL}(\infty,\mathbb{R}),\mathrm{O}(\infty)\bigr)$	$\bigl(\mathrm{U}(\infty),\mathrm{O}(\infty)\bigr)$
$\bigl(\mathrm{GL}(\infty,\mathbb{C}),\mathrm{U}(\infty)\bigr)$	$\bigl(\mathrm{U}(\infty)\times\mathrm{U}(\infty),\mathrm{U}(\infty)\bigr)$
$\bigl(\mathrm{GL}(\infty,\mathbb{H}),\mathrm{Sp}(\infty)\bigr)$	$\bigl(\mathrm{U}(2\infty),\mathrm{Sp}(\infty)\bigr)$
$\bigl(\mathrm{U}(\infty,\infty),\mathrm{U}(\infty)\times\mathrm{U}(\infty)\bigr)$	$\bigl(\mathrm{U}(2\infty),\mathrm{U}(\infty)\times\mathrm{U}(\infty)\bigr)$
$\bigl(\mathrm{Sp}(2\infty,\mathbb{R}),\mathrm{U}(\infty)\bigr)$	$\bigl(\mathrm{Sp}(\infty),\mathrm{U}(\infty)\bigr)$
$\bigl(\mathrm{Sp}(2\infty,\mathbb{C}),\mathrm{Sp}(\infty)\bigr)$	$\bigl(\mathrm{Sp}(\infty)\times\mathrm{Sp}(\infty),\mathrm{Sp}(\infty)\bigr)$
$\bigl(\mathrm{Sp}(\infty,\infty),\mathrm{Sp}(\infty)\times\mathrm{Sp}(\infty)\bigr)$	$\bigl(\mathrm{Sp}(2\infty),\mathrm{Sp}(\infty)\times\mathrm{Sp}(\infty)\bigr)$
$\bigl(\mathrm{O}(\infty,\infty),\mathrm{O}(\infty)\times\mathrm{O}(\infty)\bigr)$	$\bigl(\mathrm{O}(2\infty),\mathrm{O}(\infty)\times\mathrm{O}(\infty)\bigr)$
$\bigl(\mathrm{O}(\infty,\mathbb{C}),\mathrm{O}(\infty),\bigr)$	$\bigl(\mathrm{O}(\infty)\times\mathrm{O}(\infty),\mathrm{O}(\infty)\bigr)$
$\bigl(\mathrm{SO}^*(2\infty),\mathrm{U}(\infty)\bigr)$	$\bigl(\mathrm{O}(2\infty),\mathrm{U}(\infty)\bigr)$

Also there are 3 series of symmetric pairs of finite rank:

$\bigl(\mathrm{U}(p,\infty),\mathrm{U}(p)\times\mathrm{U}(\infty)\bigr)$	$\bigl(\mathrm{U}(p+\infty),\mathrm{U}(p)\times\mathrm{U}(\infty)\bigr)$
$\bigl(\mathrm{O}(p,\infty),\mathrm{O}(p)\times\mathrm{O}(\infty)\bigr)$	$\bigl(\mathrm{O}(p+\infty),\mathrm{O}(p)\times\mathrm{O}(\infty)\bigr)$
$\bigl(\mathrm{Sp}(p,\infty),\mathrm{Sp}(p)\times\mathrm{Sp}(\infty)\bigr)$	$\bigl(\mathrm{Sp}(p+\infty),\mathrm{Sp}(p)\times\mathrm{Sp}(\infty)\bigr)$

In pairs (G,K) of the left column, the group G ranges in all irreducible *infinite-dimensional classical groups.*[3]. In the first table in the first column, the subgroup K is *a maximal heavy subgroup* in G. The maximal heavy subgroup in $\mathrm{U}(p,\infty)$ is $\mathrm{U}(\infty)$.

Sphericity theorem. *All the above listed pairs (G,K) are spherical.*

Spherical representations are classified, see [31], [24], [25], [28], spherical functions are given by simple explicit expressions. For instance, consider the pair $\bigl(\mathrm{GL}(\infty,\mathbb{R}),\mathrm{O}(\infty)\bigr)$. Spherical representations are determined by finite collections of real parameters, $s_1, \dots, s_p$ and also $a\in\mathbb{R}$, $\sigma=0,1$. Spherical functions are given by

$$\Phi_{s,a,\sigma}(g)=|\det g|^{ia}\cdot\mathrm{sgn}\cdot(\det g)^{\sigma}\prod_k\prod_l\left(\frac{1+is_k}{2}\lambda_l+\frac{1-is_k}{2}\lambda_l^{-1}\right)^{-1/2},$$

where λ_j denote the singular numbers of $g\in\mathrm{GL}(\infty,\mathbb{R})$.

Remark. For the pair $\bigl(\mathrm{U}(\infty)\times\mathrm{U}(\infty),\mathrm{U}(\infty)\bigr)$ a substantial harmonic analysis now is constructed, see [1], apparently it exists for other pairs of the second column, see [16]. □

The class of spherical representations is not closed with respect to natural operations (tensor products, restriction to subgroups), it is reasonable to extend it.

[3]Of course, the groups $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$, $\mathrm{Sp}(\infty)$ also are classical, $p=0$.

1.5. Admissible representations. Fix a pair (G, K) from the list. We say that a unitary representation of G *admissible* ([25]), if it is continuous on the heavy subgroup K. We also use as a synonym the term "*representations of the pair* (G, K)". Certainly, it is possible to define admissibility in terms of appropriate topologies on G (see a discussion in [15]).

Lemma 1.2 *Spherical representations are admissible.*

PROOF. It is easy to show that the subspace $\cup_\alpha H^{[\alpha]}$ is invariant with respect to G. On the other hand it is non-zero (it contains the spherical vector). In virtue of irreducibility, it is dense. $\square$

We emphasis that groups of finite matrices $\mathrm{GL}(\infty, \mathbb{R})$, $\mathrm{U}(\infty)$, $\mathrm{O}(\infty, \mathbb{C})$ etc. are not type[4] I groups, and theory of their unitary representations in the usual sense is impossible. The class of admissible representations is observable and substantial works on representations of classical infinite-dimensional groups usually can be included to this scheme.

REMARK. The list contains pairs $\bigl(\mathrm{O}(2\infty), \mathrm{O}(\infty) \times \mathrm{O}(\infty)\bigr)$, $\bigl(\mathrm{O}(2\infty), \mathrm{U}(\infty)\bigr)$. The group G in both cases is the same, but conditions of continuity with respect to K are different. Therefore we get two different classes of representations. $\square$

REMARK. The list contains the pair $(\mathrm{U}(\infty) \times \mathrm{U}(\infty), \mathrm{U}(\infty))$. Its representations are not tensor products $\rho_1 \otimes \rho_2$ of representations of $\mathrm{U}(\infty)$ (the corresponding finite-dimensional theorem, see [5], 13.11.8, in this case is not valid), on representations of this pair, see [30], [31], [25]. $\square$

1.6. Product of double cosets and train. Let ρ be an admissible representation of the pair (G, K) in the space H. As above, let $H^{[\alpha]}$ be spaces of K^α-fixed vectors. Denote by $P^{[\alpha]}$ the operator of orthogonal projection to $H^{[\alpha]}$. We define operators

$$\overline{\rho}_{\alpha,\beta}(g) = P^{[\beta]} \rho(g) : H^{[\alpha]} \to H^{[\beta]}.$$

It easy to verify that for any $h_1 \in K^\alpha$, $h_2 \in K^\beta$ we have

$$\overline{\rho}_{\alpha,\beta}(h_2 g h_1) = \overline{\rho}_{\alpha,\beta}(g).$$

Therefore the function $\overline{\rho}_{\alpha,\beta}$ is defined on double cosets $\mathfrak{g} \in K^\beta \setminus G/K^\alpha$.

Multiplicativity theorem. *There exists a natural multiplication* $(\mathfrak{g}, \mathfrak{h}) \mapsto \mathfrak{g} \circ \mathfrak{h}$,

$$K^\gamma \setminus G/K^\beta \; \times \; K^\beta \setminus G/K^\alpha \; \to \; K^\gamma \setminus G/K^\alpha,$$

defined for all α, β, $\gamma \in \mathbb{Z}_+$. *Moreover, for any admissible unitary representation* ρ *of the pair* (G, K) *the following equality holds*

$$\overline{\rho}_{\beta,\gamma}(\mathfrak{g}) \overline{\rho}_{\alpha,\beta}(\mathfrak{h}) = \overline{\rho}_{\alpha,\gamma}(\mathfrak{g} \circ \mathfrak{h}). \tag{1.1}$$

[4] For definition, see, e.g., [5].

We get a category, we call it by *train* $\mathfrak{T}(G,K)$ of the pair (G,K). Objects of the train are indices (multi-indices) $\alpha \geqslant 0$, and morphisms are double cosets. The formula (1.1) claims that we get a representation of a category (see [15], II.5), i.e., a functor from $\mathfrak{T}(G,K)$ to the category of Hilbert spaces and linear bounded operators.

The operation $g \mapsto g^{-1}$ induces the map $K^\alpha \setminus G/K^\beta \to K^\beta \setminus G/K^\alpha$, we denote it by $\mathfrak{g} \mapsto \mathfrak{g}^*$. It satisfies the identity

$$(\mathfrak{g} \circ \mathfrak{h})^* = \mathfrak{h}^* \circ \mathfrak{g}^*.$$

The representation $\overline{\rho}$ is a $*$-*representation* in the following sense:

$$\rho(\mathfrak{g}^*) = \rho(\mathfrak{g}).$$

Approximation property. *Let τ be a representation of the category $\mathfrak{T}(G,K)$. Let $\|\tau(\mathfrak{g})\| \leqslant 1$ for all $\mathfrak{g}$, $\tau(1) = 1$. Then the representation τ has the form $\overline{\rho}$ for uniquely defined representation ρ of the pair (G,K).*

1.7. Comparison with finite-dimensional case. Let G be a Lie group, L a compact subgroup, not necessarily maximal. Denote by $\mathcal{M}(L \setminus G/L)$ the space of finite measures that are invariant with respect to left and right shifts by elements of L. Evidently, $\mathcal{M}(L \setminus G/L)$ is an algebra with respect to the convolution. For instance, if G is a p-adic $\mathrm{GL}(n)$ and L iwahoric subgroup, we get the Hecke algebra (see. [12]); if G is a real Lie group of rank 1 and L the maximal compact subgroup, then we get the hypergroup of generalized translate, see [9].

Let ρ be a unitary representation of G in the space H. Denote by H^L the space of L-fixed vectors. For $\mu \in \mathcal{M}(L \setminus G/L)$ the operator $\rho(\mu)$ has the block form $\begin{pmatrix} A(\mu) & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition $H = H^L \oplus (H^L)^\perp$. Thus we get a functor from unitary representations of G to representations of the convolution algebra $\mathcal{M}(L \setminus G/L)$.

It turns out that for infinite-dimensional groups the convolution of double cosets can degenerate to a multiplication. A first example. apparently was discovered by R.S.Ismagilov [6], [7], further [21], [23], [25], [14], numerous examples are discussed in [15].

1.8. Explicit formula for the product. See [25]. For definiteness, consider the pair $(G,K) = (\mathrm{GL}(\infty,\mathbb{R}), \mathrm{O}(\infty))$ and double cosets $K^\beta \setminus G/K^\alpha$. In other words, we consider matrices $\mathfrak{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of size $(\beta+\infty)\times(\alpha+\infty)$ up to the equivalence

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} 1_\beta & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & V \end{pmatrix}, \tag{1.2}$$

where U, V are orthogonal matrices, 1_α denotes the unit matrix of size α. The product in the category $\mathfrak{T}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \begin{pmatrix} P & Q \\ R & T \end{pmatrix} := \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P & 0 & Q \\ 0 & 1 & 0 \\ R & 0 & T \end{pmatrix} = \begin{pmatrix} AP & B & AQ \\ CP & D & CQ \\ R & 0 & T \end{pmatrix} \tag{1.3}$$

Note that a similar (but not precisely the same) algebraic structure is known as a product of operator colligations, see, e.g., [3].

1.9. Characteristic functions (below we do not develop this topic). It appears (see [15], IX.4), that $\circ$-multiplication can be transformed to a more customary operation[5] in the following way. Fix $\lambda \in \mathbb{C} \cup \infty$. Write the equation

$$\begin{pmatrix} p_+ \\ \lambda x \\ p_- \\ x \end{pmatrix} = \begin{pmatrix} \begin{matrix} A & B \\ C & D \end{matrix} & 0 \\ 0 & \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} q_+ \\ y \\ q_- \\ \lambda y \end{pmatrix}.$$

Consider all vectors $(p_+, p_-) \oplus (q_+, q_-) \in \mathbb{C}^{2\beta} \oplus \mathbb{C}^{2\alpha}$, for which this equation has a solution as an equation with indeterminates x, y. We get a subspace $\chi(\lambda)$ of half-dimension in $\mathbb{C}^{2\beta} \oplus \mathbb{C}^{2\alpha}$ depending on λ. We regard it as an relation (a multi-valued map) from $\mathbb{C}^{2\alpha}$ to $\mathbb{C}^{2\beta}$.

On this language, $\circ$-multiplication corresponds to the product of characteristic functions, i.e., pointwise product of relations.

Note that $\chi(\lambda)$ is a rational map from Riemann sphere to the Lagrangian Grassmannian in $\mathbb{C}^{2\alpha} \oplus \mathbb{C}^{2\beta}$. The function $\chi(\lambda)$ is also J-contractive in the Potapov sense [29].

1.10. Self-similarity. See. [26]. Let $(G, K) = \bigl(\mathrm{GL}(\infty, \mathbb{R}), \mathrm{O}(\infty)\bigr)$. Denote by $G^\alpha \subset G$ the stabilizer of initial α basis vectors in ℓ_2. Then the pair (G^α, K^α) is isomorphic to (G, K). Similar pairs are defined in all the cases, but a definition requires some care: we take basis vectors fixed by K^α, then G^α is their stabilizer in G.

Fix an unitary representation ρ of (G, K). Its restriction to a sufficiently small subgroup $(G^\alpha, K^\alpha) \simeq (G, K)$ contains a spherical subrepresentation. All spherical representations of (G, K) obtained in this way are equivalent (for all α).

1.11. Mantle. Let $(G, K) = \bigl(\mathrm{GL}(\infty, \mathbb{R}), \mathrm{O}(\infty)\bigr)$. Split $\mathbb{N}$ into two countable subsets $\mathbb{N} = \Xi \cup \Omega$ (for instance, odd and even numbers). Denote by $G^\Xi \subset G$ (respectively $G(\Xi)$) the stabilizer of basis vectors enumerated by elements of Ξ (respectively, Ω). In other words, G^Ξ and $G(\Xi)$ consist of matrices of the forms $\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ and $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, where blocks correspond to the decomposition $\ell_2(\mathbb{N}) = \ell_2(\Xi) \oplus \ell_2(\Omega)$. Evidently, $\bigl(G(\Xi), K(\Xi)\bigr)$ is isomorphic to (G, K). We fix the isomorphism

$$i_\tau : (G, K) \to \bigl(G(\Xi), K(\Xi)\bigr),$$

induced by some bijection $\tau : \mathbb{N} \to \Xi$.

[5]This is an analog of Livshits characteristic function, which is a tool in spectral theory of non-self-adjoint operators, see [11], [29], [3]. The Livshits function also take part in the representation theory of the pair $\bigl(\mathrm{U}(1+\infty) \times \mathrm{U}(\infty), \mathrm{U}(\infty)\bigr)$, see. [27].

Let ρ be a unitary representation of (G, K) in the space H. Denote by $H^{[\Xi]}$ the space of K^{Ξ}-fixed vectors. By $P^{[\Xi]}$ we denote the operator of projection on $H^{[\Xi]}$.

Lemma about self-restriction. *The space* $H^{[\Xi]}$ *is* $G(\Xi)$*-invariant. The representation* $\rho \circ i_\tau$ *of the pair* (G, K) *in* $H^{[\Xi]}$ *is equivalent to* ρ.

On the other hand, for each $g \in G$ we assign the operator $\overline{\rho}_{\Xi,\Xi}(g) : H^{[\Xi]} \to H^{[\Xi]}$ as

$$\overline{\rho}_{\Xi,\Xi}(g) = P^{[\Xi]}\rho(g).$$

As above, $\overline{\rho}$ is a function on double cosets.

Version of multiplicativity theorem. *Double cosets* $\Gamma_\infty = K^\Xi \setminus G/K^\Xi$ *admit a natural structure of a semigroup* (**mantle** *of the pair* (G, K))*, the map* $\overline{\rho}_{\Xi,\Xi}$ *is a representation of* Γ_∞.

Product is given by the same formula (1.3), in the present situation all blocks of the matrix are infinite.

Next, notice that the natural map $G(\Xi) \simeq G \to K^\Xi \setminus G/K^\Xi$ is injective, therefore we observe that the representation of the semigroup $K^\Xi \setminus G/K^\Xi$ extends the representation ρ of $G(\Xi) \simeq G$.

As a result, we get

Theorem about mantle. *Any unitary representation of the pair* (G, K) *can be extended canonically to a representation of the semigroup* Γ_∞.

1.12. Spherical characters. Notice, that $Z := K^\Xi\backslash G^\Xi/K^\Xi \subset K^\Xi\backslash G/K^\Xi$ is a central subsemigroup in the mantle. In any irreducible representation ρ of the pair (G, K) it acts by scalar operators. This determines a multiplicative character $\Phi_\rho : Z \to \mathbb{C}^\times$.

If ρ is spherical, then Φ_ρ coincides with the spherical function of ρ. Generally, it coincides with spherical functions of subgroups as in 1.10.

1.13. Non-standard pairs. Until this moment we discussed pairs (G, K) containing in the Olshanski list. Now let $G = \mathrm{GL}(\infty, \mathbb{R}) \times \dots \times \mathrm{GL}(\infty, \mathbb{R})$, let $K \subset G$ be the group $\mathrm{O}(\infty)$ embedded to G by the diagonal.

a) The pair (G, K) is spherical. Notice, that this statement has no finite-dimensional analogs. Multiplicativity theorem also holds.

b) Nessonov [19], [20] obtained description of all spherical representations of this pair.

c) The paper [18] contains a construction of characteristic functions, it is valid in our case.

There arises the following question: In which generality the statements of Subsections 1.4–1.12 hold?

1.14. Purposes of the paper. We introduce class of (G, L)*-pairs*, where G is a classical group, L is a heavy subgroup. For this class the following hold true:

— the multiplicativity theorem;

— the approximation property;

— the construction of the mantle;

— existence of spherical characters.

The construction of characteristic functions given in [18] survives in these cases. (in the present paper this topic is not discussed).

Also, we define a subclass of *pure* (G, L)*-pairs*. In this case there hold
— sphericity theorem;
— self-similarity.

Definition of (G, L)-pairs is contained in Section 2, sketches of proofs in Section 3, Section 4 contains conjectures and additional remarks. We do not repeate formulations of theorems.

2 Definition

2.1. Straight embeddings of heavy groups.. First, let heavy groups L, K be irreducible, i.e, they have the form $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$, $\mathrm{Sp}(\infty)$. There are obvious homomorphisms:

$$\begin{array}{lll} \mathrm{O}(\infty)\to\mathrm{U}(\infty) & \mathrm{O}(\infty)\to\mathrm{Sp}(\infty) & \mathrm{U}(\infty)\to\mathrm{Sp}(\infty) \\ \mathrm{U}(\infty)\to\mathrm{O}(2\infty) & \mathrm{Sp}(\infty)\to\mathrm{O}(4\infty) & \mathrm{Sp}(\infty)\to\mathrm{U}(2\infty). \end{array}$$

Also element-wise complex conjugation determines a homomorphism $\mathrm{U}(\infty)$ to itself. We call such homomorphisms and also identical homomorphisms $g \mapsto g$ *trivial*.

Next, let K be irreducible, $L = L_1 \times \dots \times L_p$. We say that a homomorphism $L \to K$ is *straight*, if it has the form

$$(g_1, \dots, g_p) \mapsto \begin{pmatrix} 1_a & 0 & \dots & 0 \\ 0 & \tau_1(g_{m_1}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau_N(g_{m_N}) \end{pmatrix}, \qquad g_k \in L_k, \tag{2.1}$$

where τ_j are trivial homomorphisms. We admit unit matrix 1_a of arbitrary order $0, 1, 2, \dots, \infty$. The number N can be in the limits $0 < N < \infty$. We do not require a homomorphism to be faithful.

Finally, let $L = L_1 \times \dots \times L_p$, $K = K_1 \times \dots \times K_q$. Then a *straight homomorphism* is a product of straight homomorphisms $L \to K_1, \dots, L \to K_q$.

2.2. (G, L)**-pairs.** We say that a *classical group* $G = G_1 \times \dots \times G_r$ is a finite product of irreducible classical groups (see the list above). Let $K = K_1 \times \dots \times K_q$ be the maximal heavy subgroup in G. Let L be a straightly embedded heavy subgroup in K. We call such objects (G, L)*-pairs*.

A pair (G, L) is *pure*, if the centralizer of the subgroup L in G is trivial[6].

[6] Examples. The pair $(G, L) = \big(\mathrm{GL}(3\infty, \mathbb{R}), \mathrm{O}(\infty)\times\mathrm{O}(\infty)\times\mathrm{O}(\infty)\big)$ is pure. The centralizer of L in the group of all bounded operators is $(\mathbb{R}^\times)^3$. But it is not contained in the group $G = \mathrm{GL}(3\infty, \mathbb{R})$ of finite matrices. The pair $\big(\mathrm{O}(1 + 3\infty, \mathbb{R}), \mathrm{O}(\infty) \times \mathrm{O}(\infty) \times \mathrm{O}(\infty)\big)$ is not pure, the centralizer in $\mathrm{O}(1 + 3\infty, \mathbb{R})$ is $\mathbb{Z}_2$.

An equivalent condition: in matrices (2.1) the unit block 1_a is absent and there are no groups $\mathrm{U}(p,\infty)$, $\mathrm{O}(p,\infty)$, $\mathrm{Sp}(p,\infty)$ among factors G_j[7].

We say that a pair is *finite*, if blocks 1_a in formula (2.1) for all factors K_i have finite sizes. Finite pairs will appear only in conjectures in the last section.

2.3. Definition of multiplication on double cosets. Let L be irreducible. Fix $\alpha \geqslant 0$ and consider the sequence $\Theta_m^{[\alpha]} \in L$ given by

$$\rho(\Theta_m^{[\alpha]}) = \begin{pmatrix} 1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 \\ 0 & 1_m & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}. \tag{2.2}$$

If $L = L_1 \times \dots \times L_p$ is reducible, we fix an multi-index α and take the net depending on a multi-index m,

$$\Theta_m^{[\alpha]} = \bigl(\Theta_{m_1}^{[\alpha_1]}, \dots \Theta_{m_p}^{[\alpha_p]}\bigr).$$

Let

$$\mathfrak{g} \in K^{[\gamma]} \setminus G/K^{[\beta]}, \qquad \mathfrak{h} \in K^{[\beta]} \setminus G/K^{[\alpha]}.$$

Choose representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$.

Consider subsequence

$$z_m = g\Theta_m^{[\beta]} h \in G.$$

Proposition 2.1 *a) A double coset $\mathfrak{z}_m = K^{[\gamma]} z_m K^{[\alpha]}$ is eventually constant.*

b) The result $g \circ h = \lim \mathfrak{z}_m$ does not depend on a choice of representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$.

c) The operation obtained in this way is associative, i.e., for each α, β, γ, δ and any

$$\mathfrak{f} \in K^\delta \setminus G/K^\gamma, \qquad \mathfrak{g} \in K^\gamma \setminus G/K^\beta, \qquad \mathfrak{h} \in K^\beta \setminus G/K^\alpha$$

we have $(\mathfrak{f} \circ \mathfrak{g}) \circ \mathfrak{h} = \mathfrak{f} \circ (\mathfrak{g} \circ \mathfrak{h})$.

As a result we get a category, we call it *train* of the pair (G, L) and denote by $\mathfrak{T}(G, L)$.

Instead of formal proof in general case, we present examples explaining why this happens.

2.4. Example. Consider the pair $\bigl(\mathrm{GL}(\infty,\mathbb{R}), \mathrm{O}(\infty)\bigr)$. To simplify notation, assume $\gamma = \beta = \alpha$. Fix g, $h \in G$. Write g, h as block matrices of the size $\alpha + N + \infty$. Let N be sufficiently large. Then g, h have the form

$$g = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \qquad h = \begin{pmatrix} p & q & 0 \\ r & t & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}. \tag{2.3}$$

[7] This limitation is necessary, because for such groups the maximal heavy subgroup is not a precise analog of the maximal compact group.

The product $g\Theta^{[\alpha]}_{N+k}h$ equals

$$\begin{pmatrix} ap & aq & 0 & b & 0 & 0 \\ cp & cq & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_k & 0 \\ r & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$

Since we examine double cosets, we transpose rows and columns whose numbers $> \alpha$. In this way we can reduce our matrix to arbitrary of two convenient forms:

$$S_1 = \begin{pmatrix} ap & aq & b & 0 \\ cp & cq & d & 0 \\ r & t & 0 & 0 \\ 0 & 0 & 0 & 1_{2k+\infty} \end{pmatrix} \qquad S_2 = \begin{pmatrix} ap & b & aq & 0 \\ cp & d & cq & 0 \\ r & 0 & t & 0 \\ 0 & 0 & 0 & 1_{2k+\infty} \end{pmatrix}.$$

We observe that the result is independent on k. Next, we must verify independence on a choice of representatives, i.e., gu, vh c u, $v \in K_\alpha$ produce the same double coset. Without loss of generality, we can assume u, $v \in \mathrm{O}(N)$ (otherwise we choose larger N earlier). We replace g, h from (2.3) to

$$g = \begin{pmatrix} a & bu & 0 \\ c & du & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} p & q & 0 \\ vr & vt & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

with orthogonal matrices u, v. The new matrix S_2 is

$$\begin{pmatrix} ap & bu & aq & 0 \\ cp & du & cq & 0 \\ vr & 0 & vt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & v & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ap & b & aq & 0 \\ cp & d & cq & 0 \\ r & 0 & t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & u & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Associativity is clear from the formula (1.3).

2.5. Two examples. a) Consider the pair $G = \mathrm{GL}(\infty, \mathbb{R}) \times \cdots \times \mathrm{GL}(\infty, \mathbb{R})$ with the diagonal subgroup $L = \mathrm{O}(\infty)$. We have a finite collection of matrices $g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ defined up to the equivalence

$$\left\{ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix} \right\} \sim$$
$$\sim \left\{ \begin{pmatrix} 1_\beta & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & v \end{pmatrix}, \dots, \begin{pmatrix} 1_\beta & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & v \end{pmatrix} \right\}.$$

Then $\circ$-product is defined component-wise by the formula (1.3).

b) Let $G = \mathrm{U}(2\infty)$, $K = \mathrm{O}(\infty) \times \mathrm{O}(\infty)$. Now we have unitary matrices g determined up to the equivalence

$$\begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ c_{11} & d_{11} & c_{12} & d_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ c_{21} & d_{21} & c_{22} & d_{22} \end{pmatrix} \sim \begin{pmatrix} 1 & & & \\ & u_1 & & \\ & & 1 & \\ & & & u_2 \end{pmatrix} \begin{pmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ c_{11} & d_{11} & c_{12} & d_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ c_{21} & d_{21} & c_{22} & d_{22} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & v_1 & & \\ & & 1 & \\ & & & v_2 \end{pmatrix},$$

where u_1, u_2, v_1, $v_2 \in \mathrm{O}(\infty)$.

The product is given by

$$g \circ g' = \begin{pmatrix} a_{11} & b_{11} & 0 & a_{12} & b_{12} & 0 \\ c_{11} & d_{11} & 0 & c_{12} & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a_{21} & b_{21} & 0 & a_{22} & b_{22} & 0 \\ c_{21} & d_{21} & 0 & c_{22} & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a'_{11} & 0 & b'_{11} & a'_{12} & 0 & b'_{12} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ c'_{11} & 0 & d'_{11} & c'_{12} & 0 & d'_{12} \\ a'_{21} & 0 & b'_{21} & a'_{22} & 0 & b'_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c'_{21} & 0 & d'_{21} & c'_{22} & 0 & d'_{22} \end{pmatrix}. \qquad (2.5)$$

Double cosets are taken with respect to the group of matrices of the form

$$\begin{pmatrix} 1 & & & & & \\ & u_{11} & u_{12} & & & \\ & u_{21} & u_{22} & & & \\ & & & 1 & & \\ & & & & v_{11} & v_{12} \\ & & & & v_{21} & v_{22} \end{pmatrix}, \qquad \text{where } \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in \mathrm{O}(2\infty).$$

3 Proofs

3.1. Multiplicativity (the analog of 1.6).

Lemma 3.1 *For any unitary representation π of a heavy group L the sequence $\pi(\Theta_m^{[\alpha]})$ (see (2.2)) weakly converges to the projector $P^{[\alpha]}$ to $H^{[\alpha]}$.*

See [15], Theorem VIII.1.4. A priory proof is given for a symmetric group $S(\infty)$, proofs for $\mathrm{O}(\infty)$, $\mathrm{U}(\infty)$, $\mathrm{Sp}(\infty)$ are the same. On the other hand, it is easy to reduce the statement from the explicit classification of representations of heavy groups [23], see also [15]. $\square$

Let us prove the multiplicativity theorem for an arbitrary pair (G, L). Let $\mathfrak{g}$, $\mathfrak{h}$, g, h be the same as in Subsection 2.3. Then

$$\overline{\rho}(\mathfrak{g} \circ \mathfrak{h}) = P^{[\gamma]} \rho(g \Theta_m^{[\alpha]} h) : H^{[\alpha]} \to H^{[\gamma]}$$

for sufficiently large m. On the other hand,

$$\lim_{m \to \infty} P^{[\gamma]} \rho(g \Theta_m^{[\alpha]} h) = \lim_{m \to \infty} P^{[\gamma]} \rho(g) \rho(\Theta_m^{[\alpha]}) \rho(h) =$$
$$= P^{[\gamma]} \rho(g) \big(\lim_{m \to \infty} \rho(\Theta_m^{[\alpha]}) \big) \rho(h) = P^{[\gamma]} \rho(g) P^{[\beta]} \rho(h) = \overline{\rho}(\mathfrak{g}) \overline{\rho}(\mathfrak{h}). \qquad \square$$

Thus, for any unitary representation of (G, L) we construct a representation of the category $\mathfrak{T}(G, L)$. Now we present the inverse construction.

3.2. Approximation property. Let $\alpha < \beta$ (therefore, $L^\alpha \supset L^\beta$). Denote by $\lambda_{\alpha,\beta}$ the double coset $L^\beta \cdot 1 \cdot L^\alpha$. Set $\mu_{\beta,\alpha} := \lambda^*_{\alpha,\beta}$. Then the following identities hold:

$$\mu_{\beta,\alpha} \circ \lambda_{\alpha,\beta} = 1^{[\alpha]}, \qquad \lambda_{\beta,\gamma}\lambda_{\alpha,\beta} = \lambda_{\alpha,\gamma}, \tag{3.1}$$

where $1^{[\alpha]}$ denotes the unit $L^\alpha \cdot 1 \cdot L^\alpha$ of the semigroup $L^\alpha \setminus G/L^\alpha$. We get a structure of *ordered category* in the sense of [15], III.4.

Notice that $\psi_{\alpha,\beta} := \lambda_{\alpha,\beta} \circ \mu_{\beta,\alpha}$ as a subset in G coincides with L^α. But it is a nontrivial idempotent in $L^\beta \setminus G/L^\beta$:

$$\psi^2_{\alpha,\beta} = \psi_{\alpha,\beta} \qquad \psi^* = \psi_{\alpha,\beta}.$$

Consider a $*$-representation R of the train $\mathfrak{T}(G, L)$ by contractive operators, let R assigns unit operators to units of semigroups of endomorphisms. Denote by $H^{[\alpha]}$ the Hilbert space, corresponding to a (multi)index α. In virtue of (3.1), an operator $R(\lambda_{\alpha,\beta})$ is an operator of isometric embedding $H^{[\alpha]} \to H^{[\beta]}$. The projection operator to the image of the embedding is $\psi_{\alpha,\beta}$. Now we have a chain of isometric embedding (to be definite, let $\alpha = (\alpha_1, \alpha_2)$ be a bi-index):

$$\begin{array}{ccccccccc}
\dots & & & & H^{\alpha_1+1,\alpha_2} & & & & \dots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\dots & & H^{\alpha_1,\alpha_2} & & & & H^{\alpha_1+1,\alpha_2+1} & & \dots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\dots & & & & H^{\alpha_1,\alpha_2+1} & & & & \dots
\end{array}$$

Denote by H the limit space.

The group G is a union of finite-dimensional groups $G(\alpha)$. Notice that the natural map $G(\alpha)$ to $L^\alpha \setminus G/L^\alpha$ is an isomorphism. Therefore we get representation of $G(\alpha)$ in the space $H^{[\alpha]}$. Operators of representation are contractive with their inverses. Therefore they are unitary.

Hence the group G acts in H. We omit a watching of the remaining details. $\square$

3.3. Sphericity.

Lemma 3.2 *For a pure (G, L)-pairs the semigroup $L \setminus G/L$ is commutative.*

The case is that product in such $L \setminus G/L$ means that we "join" one matrix to another. For instance, formula (2.5) in this case is reduced to

$$g \circ g' = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \circ \begin{pmatrix} d'_{11} & d'_{12} \\ d'_{21} & d'_{22} \end{pmatrix} = \begin{pmatrix} d_{11} & 0 & d_{12} & 0 \\ 0 & d'_{11} & 0 & d'_{12} \\ d_{21} & 0 & d_{22} & 0 \\ 0 & d'_{21} & 0 & d'_{22} \end{pmatrix}.$$

To get $g' \circ g$, we conjugate right-hand side by a matrix

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \qquad \text{where} \qquad J = \begin{pmatrix} 0 & 0 & 1_N & 0 \\ 0 & 1_\infty & 0 & \\ 1_N & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}$$

with sufficiently large N. $\square$

Thus $L \setminus G/L$ is an Abelian semigroup with involution. Its irreducible representations are one-dimensional. On the other hand, irreducibility of representation of (G, L) is equivalent to irreducibility of the corresponding representation of the train $\mathfrak{T}(G, L)$; but for an irreducible representation of an ordered category, all representations of semigroups of endomorphisms are irreducible (see [15], Lemma III.4.3). This implies also the following statement.

Proposition 3.3 *Spherical functions of the pair (G, L) are homomorphisms*[8] *from the semigroup $L \setminus G/L$ to the multiplicative group $\mathbb{C}^\times$.*

3.4. Self-similarity. Consider a pure pair (G, L) and its subgroup G^α as in 1.8.

Lemma 3.4 *The image of G^α in $L^\alpha \setminus G/L^\alpha$ is a central semigroup canonically isomorphic to $L \setminus G/L$.*

Proof. The isomorphism is induced by the "shift" $G \to G^\alpha$, i.e., $g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$. Further, let $g \in G$, $h \in G^\alpha$. It is easy to see that for sufficiently large m,

$$g\,(\Theta_m^{[\alpha]} h \Theta_m^{[\alpha]}) = (\Theta_m^{[\alpha]} h \Theta_m^{[\alpha]})\, g \quad \Rightarrow \quad L^\alpha \cdot g\Theta_m^{[\alpha]} h \cdot L^\alpha = L^\alpha \cdot h\Theta_m^{[\alpha]} g \cdot L^\alpha, \qquad \square$$

Next, for $\alpha < \beta$ there is a natural map $K^\beta \setminus G/K^\beta \to K^\alpha \setminus G/K^\alpha$ that can be defined by the formula

$$\Pi : \mathfrak{g} \mapsto \mu_{\beta,\alpha} \circ \mathfrak{g} \circ \lambda_{\alpha,\beta}.$$

Lemma 3.5 *The projection $\Pi : K^\beta \setminus G^\beta/K^\beta \to K^\alpha \setminus G^\alpha/K^\alpha$ of central semigroups coincides with isomorphism induced by the shift $G^\beta \to G^\alpha$.*

Proof. The element $g \in G^\alpha$ and the corresponding element $\widetilde{g} \in G^\beta$ are conjugate in G^α by an element of the group K^α. $\square$

Therefore the semigroup $L \setminus G/L$ acts in the whole space H. If a representation ρ is irreducible, then this semigroup acts by scalar operators. The corresponding character coincides with spherical functions of groups (G^α, K^α) acting in H.

3.5. Self-restriction. Let us prove the statement for $(G, L) = \big(\mathrm{GL}(\infty, \mathbb{R}), \mathrm{O}(\infty)\big)$. We preserve the notation of 1.11.

[8] non-arbitrary.

The map $\tau : \mathbb{N} \to \Xi$ determines 0-1 matrix, denote it by A, the norm of A is 1. According Theorem 1.1, representations of a heavy group admits a weakly continuous extension to representation of semigroup of contructions. In particular, for any continuous representation π of L in a Hilbert space H, there is an operator $\pi(A)$ assigned to the matrix A.

Lemma 3.6 *$\pi(A)$ is an isometric map from H to the subspace $H^{[\Xi]}$.*

PROOF. First, $(\pi(A))^*\pi(A) = \pi(A^*A) = \pi(1) = 1$. Therefore $\pi(A)$ is an isometric embedding. Similarly, $\pi(A)(\pi(A))^* = \pi(AA^*)$. The operator AA^* is the projection to the subspace $\ell_2(\Xi)$ in $\ell_2(\mathbb{N})$. Next, we again refer to explicit form of representations of heavy groups, [23]. $\square$

Consider an admissible representation ρ of a pair (G, L). Denote by π the restriction of ρ to L. Extend π to the semigroup of contractions.

Lemma 3.7 *The operator $\pi(A)$ intertwines the representation π of the pair (G, L) in the space H and the representation $\pi \circ i_\tau$ in the space $H^{[\Xi]}$.*

PROOF. Consider a sequence of matrices-permutations B_m weakly converging to A. Then $\pi(B_m)$ weakly converges to $\pi(A)$. The group G is an inductive limit of reductive groups $G(n)$. For each group $G(n)$ the operator $\pi(B_m)$ with sufficiently large number m is intertwining. Therefore the limit is intertwining for the whole group G. $\square$

Let $L = L_1 \times \cdots \times L_p$. Each factor is a group of unitary operators in a certain space $\ell_1 = \ell_2(\mathbb{N})$. We split each copy of $\mathbb{N}$ into two countable subsets $\mathbb{N} = \Xi_j \cup \Omega_j$. Fix bijections $\tau_j : \mathbb{N} \to \Xi_j$. Now we can repeat the same words.

3.6. Mantle. Theorem on mantle follows from multiplicativity theorem and lemma about self-restriction.

3.7. Spherical characters, see 1.12. Consider a (G, L)-pair, acting in a direct sum of the spaces ℓ_2 as in 2.1–2.2. Consider all basis vectors e_μ in $\oplus\ell_2$ that are fixed by the group L. Consider the subgroup $G_{min} \subset G$ stabilizing all e_μ.

EXAMPLE. Let $(G, L) = \big(\mathrm{GL}(7 + 2\infty, \mathbb{R}), \mathrm{O}(\infty) \times \mathrm{O}(\infty)\big)$. Then $G_{min} = \mathrm{GL}(2\infty, \mathbb{R})$.

Proposition 3.8 *Mantle of the pair (G, L) contains a central semigroup canonically isomorphic $L \setminus G_{min}/L$.*

Explain this by an example. Let $(G, L) = \big(\mathrm{U}(p + \infty) \times \mathrm{U}(\infty), \mathrm{U}(\infty)\big)$, the subgroup $L \simeq \mathrm{U}(\infty)$ is embedded to the product by the diagonal, $G_{min} = \mathrm{U}(\infty) \times \mathrm{U}(\infty)$. Redenote (G, L) as $\big(\mathrm{U}(p + 2\infty) \times \mathrm{U}(2\infty), \mathrm{U}(2\infty)\big)$, we write its elements as pairs of block matrices (g, h), where the size of g is $(p + \infty + \infty)$ and the size of h is $(\infty + \infty)$. Consider a subgroup $Z \simeq \mathrm{U}(\infty) \times \mathrm{U}(\infty)$ consisting of pairs of matrices

$$\left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & d_1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & d_2 \end{pmatrix} \right\},$$

where $d_1, d_2 \in \mathrm{U}(\infty)$. Consider a subgroup $M \subset Z$ consisting of pairs satisfying $d_1 = d_2$. Then the mantle is the semigroup $M \setminus G/M$. It contains the central subgroup $M \setminus Z/M$. Property of centrality is a version of Lemma 3.4 $\square$

4 Discussion

4.1. General conjectures. Let a pair (G, L) be finite, see .2.2. Then

a) Unitary representations of (G, L) have type I.

b) There exists only countable set of representations with a given spherical character.

c) Let $(G, L) \to (G', L')$ be an embedding of pairs such that the embedding $L \to L'$ is straight in the sense of 2.1. Then the restriction of any irreducible representation ρ of the pair (G', L') to (G, L) has a pure discrete spectrum. In particular, this is the case for tensor products of irreducible representations.

d) An explicit classification of spherical representations is possible. Precisely, they can be obtained by the construction described in [15], IX.5 (the difference between symmetric and non-symmetric pairs is explained in IX.5.6).

e) In [18] for any element $\mathfrak{g} \in L^\alpha \setminus G/L^\beta$ there were obtained spectral data: "characteristic function" (it is an inner function of a matrix argument) and "eigensurface" (a hypersurface in a Grassmannian). The element $\mathfrak{g}$ can be uniquely reconstructed from these data.

4.2. Why we consider only such pairs? There are many embeddings of $\mathrm{U}(n)$ to unitary groups $\mathrm{U}(N)$ of big size. For instance, $g \mapsto g \otimes g$ determines an embedding $\mathrm{U}(n) \to \mathrm{U}(n^2)$. But for finite $g \in \mathrm{U}(\infty)$ the matrix $g \otimes g$ is not finite. I.e., formally, embeddings $\mathrm{U}(n)$ to $\mathrm{U}(\cdot)$, different from straight embeddings do not survive as $n \to \infty$.

Certainly, this argument is not sufficient. We can search natural subgroups in the complete operator group in $\ell_2(\mathbb{N} \times \mathbb{N})$ containing matrices $g \otimes g$. The author do not see such subgroups.[9].

4.3. Sphericity. There are limits of spherical pairs that are not pure, say $\bigl(\mathrm{U}(1+\infty) \times \mathrm{U}(\infty), \mathrm{U}(\infty)\bigr)$, $\bigl(\mathrm{O}(1+2\infty), \mathrm{U}(\infty)\bigr)$, see [27].

Also there are many cases of sphericity with respect to subgroups of the form $L \times M$, where L is heavy and M is compact (see, e.g., the pair $\mathrm{U}(p+\infty)/\mathrm{U}(p) \times \mathrm{U}(\infty)$ mentioned above).

4.4. Symmetric groups. For symmetric groups similar facts hold, an example is discussed in [17]. Our proofs are based on Lemma 3.1, they survive literally for symmetric groups.

[9]For instance, the subgroup generated by finite unitary matrices and matrices $g \otimes g$ is a direct product of two generating subgroups.

References

[1] Borodin, A., Olshanski, G. *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes.* Ann. of Math. (2) 161 (2005), no. 3, 1319–1422.

[2] Brion, M. *Classification des espaces homogènes sphériques*, Compositio Math. 63 (1987), 189–208.

[3] Brodski, M. S., *Unitary operator colligations and their characteristic functions.* Russ. Math. Surveys, 1978, 33:4, 159–191.

[4] Gelfand, I. M. *Spherical functions in symmetric Riemann spaces.* (Russian) Doklady Akad. Nauk SSSR (N.S.) 70, (1950). 5–8. English transl. in Transl., Ser. 2, Amer. Math. Soc. 37 (1964), 39–43.

[5] Dixmier, J. *Les C^*-algèbres et leurs représentations.* Gauthier Villars, 1964.

[6] Ismagilov, R.S., *Elementary spherical functions on the groups* $\mathrm{SL}(2,P)$ *over a field P, which is not locally compsct with respect to the subgroup of matrices with integral elements.* Math. USSR-Izvestiya, 1967, 1:2, 349–380.

[7] Ismagilov, R.S., *Spherical functions over a normed field whose residue field is infinite.* Funct. Anal. and Its Appl. Volume 4, N. 1, 37–45.

[8] Knop, F. *Semisymmetric polynomials and the invariant theory of matrix vector pairs.* Represent. Theory 5 (2001), 224–266.

[9] Koornwinder, T. H. *Jacobi functions and analysis on noncompact semisimple Lie groups.* In *Special functions: group theoretical aspects and applications,* 1–85, Math. Appl., Reidel, Dordrecht, 1984.

[10] M. Krämer, *Sphärische Untergruppen in kompakten zusammenheängenden Liegruppen*, Compositio Math. 38 (1979), no. 2, 129–153.

[11] Livshits, M.S. *On a certain class of linear operators in Hilbert space.* Mat. Sb., N. Ser. 19(61), 239-262 (1946), English transl. in Amer. Math. Soc. Transl. (Ser. 2), Vol. 13, 61–83 (1960)

[12] Macdonald, I. G. *Symmetric functions and Hall polynomials.* Second edition. With contributions by A. Zelevinsky. Oxford University Press, New York, 1995.

[13] Mikityuk, I. V. *On the integrability of invariant Hamiltonian systems with homogeneous configuration spaces*,Math. USSR-Sb. 57 (1987), 527–546.

[14] Neretin, Yu. A. *Categories of bistochastic measures and representations of some infinite-dimensional groups.* Russian Acad. Sci. Sb. Math. 75 (1993), no. 1, 197–219.

[15] Neretin, Yu. A. *Categories of symmetries and infinite-dimensional groups.* Oxford University Press, New York, 1996.

[16] Neretin, Yu. A. *Hua type integrals over unitary groups and over projective limits of unitary groups,* Duke Math. J. 114 (2002), 239–266.

[17] Neretin, Yu. A., *Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories.* Preprint, available via `arXiv:0909.4739`.

[18] Neretin, Yu. A. *Multi-operator colligations and multivariate spherical functions.* Preprint, available via `arXive:1006.2275`.

[19] Nessonov, N. I., *Factor-representation of the group $GL(\infty)$ and admissible representations* $\mathrm{GL}(\infty)^X$, (Russian) Math. Phys., Analysis, Geometry (Kharkov Math. Journ.), 2003, N4, 167–187.

[20] Nessonov, N.I. *Factor-representation of the group $GL(\infty)$ and admissible representations of $GL(\infty)^X$. II.* (Russian) Mat. Fiz. Anal. Geom. (Kharkov Math. Journ.) 2003, No. 4, 524–556.

[21] Olshanski, G. I. *New "large" groups of type* I. Current problems in mathematics, Vol. 16 (Russian), pp. 31–52, 228, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii (VINITI), Moscow, 1980. English transl. J. Sov. Math. 18 (1982) 22–39.

[22] Olshanski, G. I. *Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe.* Soviet Math. Dokl. 27 (1982), no. 2, 290–294.

[23] Olshanski, G.I *Infinite-dimensional groups of finite R-rank: description of representations and asymptotic theory.* Funct. Anal. and Appl., 1984, 18:1, 22–34.

[24] Olshanski, G. I. *Unitary representations of the group* $\mathrm{SO}_0(\infty,\infty)$ *as limits of unitary representations of the groups* $\mathrm{SO}_0(n,\infty)$ *as* $n \to \infty$. Funct. Anal. Appl. 20 (1986), no. 4, 292–301.

[25] Olshanski, G.I. *Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe.* In *Representation of Lie groups and related topics,* Adv. Stud. Contemp. Math. 7, 269–463 (1990).

[26] Olshanski, G.I., *Unitary representations of (G,K)-pairs connected with the infinite symmetric group $S(\infty)$.* Leningr. Math. J. 1, No.4, 983–1014 (1990).

[27] Olshanski, G.I. *Caractères généralisés du groupe $U(\infty)$ et fonctions intèrieures.* Comptes Rendus Acad. Sci. Paris. Sér. 1, 313 (1991), 9-12.

[28] Pickrell, D. *Separable representations for automorphism groups of infinite symmetric spaces.* J. Funct. Anal. 90 (1990), no. 1, 1–26.

[29] Potapov, V. P. *The multiplicative structure of J-contractive matrix functions.* Trudy Moskov. Mat. Obschestva. 4 (1955), 125–236; English transl. Amer. Math. Soc. Transl. (2) 15 (1960), 131–243.

[30] Strătilă, ., Voiculescu, D., *Representations of AF-algebras and of the group* $U(\infty)$. Lecture Notes in Mathematics, Vol. 486 (1975).

[31] Vershik, A. M.; Kerov, S. V. Characters and factor-representations of the infinite unitary group. Soviet Math. Dokl. 26, 570–574 (1983).

Math.Dept., University of Vienna,
Nordbergstrasse, 15, Vienna, Austria
&
Institute for Theoretical and Experimental Physics,
Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia
&
Mech.Math.Dept., Moscow State University,
Vorob'evy Gory, Moscow
e-mail: neretin(at) mccme.ru
URL:www.mat.univie.ac.at/∼neretin
wwwth.itep.ru/∼neretin